Lecture Note 4

Question 1

Minimize $x_1 + x_2 + x_3$

$$x_1 + x_2 \geq 1$$
$$x_1 + x_3 \geq 1$$
$$x_2 + x_3 \geq 1$$

Question 2 $\quad x_1 = x_2 = x_3 = 1/2$

Question 3 $\quad S = \{1, 2, 3\} \rightarrow$ not optimal as $S^* = \{1, 2\}$

However, it satisfied the inequality as $\underbrace{|S|}_{3} \leq 2\underbrace{|S^*|}_{2}$

Question 4 $\quad$ We will have $S = \{1, 2, 3, 4\} \rightarrow$ not optimal but $\underbrace{|S|}_{4} \leq 2 \underbrace{|S^*|}_{2}$

Question 5 $\quad$ Input : Set V and $E \subseteq \{\{u, v\} : u, v \in V\}$, k

Question 6 $\quad$ Output : $S \subseteq V$

Question 8 $\quad$ Objective Function : Minimize $|S|$

Question 7 $\quad$ Constraint : $|\{\{u, v\} \in E : u \in S \text{ or } v \in S\}| \geq |E| - k$.

Question 9 $\quad$ % return YourOptimizationModel $(V, E, 0)$;

Question 10 $\quad$ As we can solve Vertex Cover by an algorithm, this problem is not easier than Vertex Cover. ~~As this proble~~ As Vertex Cover is NP-hard, this problem is also NP-hard.

Question 11 $\quad S = \{2, 3\}$

Question 2.2 $\quad$ Minimize $x_1 + x_2 + x_3 + x_4 + x_5$

$$x_1 + x_2 - r_1 \geq 0$$
$$x_1 + x_3 - r_2 \geq 0$$
$$x_2 + x_3 - r_3 \geq 0$$
$$x_2 + x_4 - r_4 \geq 0$$
$$x_3 + x_5 - r_5 \geq 0$$
$$r_1 + r_2 + r_3 + r_4 + r_5 \geq 4$$

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 4 \end{bmatrix} \qquad c = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Question 2.3

$$A = \begin{bmatrix} \text{matrix in vertex cover} & -\text{identity matrix} \\ 0\ 0\ 0 \cdots 0 & 1\ 1 \cdots 1 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ |E|-k \end{bmatrix} \quad c = \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Question 2.4

We have $x_1 = x_2 = x_3 = 1/3$. Then, we have $r_1 = r_2 = r_3 = \frac{2}{3}$.

$$r_1 + r_2 + r_3 \geq |E| - k = 3 - 1 = 2$$

Question 2.5

We will have $x_1' = x_2' = x_3' = 0 \rightarrow S = \emptyset$

Question 2.6

$$x_i' = \begin{cases} 0 & \text{if } x_i < \frac{1}{2} \cdot \frac{|E|-k}{|E|} \\ 1 & \text{otherwise.} \end{cases}$$

There will be at least $|E|-k$ members of $\{r_1, \ldots, r_{|E|}\}$ that $\geq \frac{|E|-k}{|E|} \cdot r_i$

Lecture Note 3

Q1 : Set V, Set $E \subseteq \{\{u,v\} : u, v \in V\}$

maximum # courses that we can recommend to a particular person k.

Q2 : For each $e \in E$, # courses recommend to e, x_e

Q3 : Maximize $\sum_e x_e$

Q4 : For all v, $\sum_{e: v \in e} x_e \le k$.

Q5 : A is an incident matrix of V, E, $b = \begin{bmatrix} k \\ k \\ \vdots \\ k \end{bmatrix}$, $c = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$

We then have Maximize $c^T x$
s.t. $Ax \le b$.

Q6 : Solve linear program in Q5 to have $\begin{bmatrix} x_1 \\ \vdots \\ x_{|E|} \end{bmatrix}$

Have $x_i' = \begin{cases} k & \text{with prob. } x_i / 2|V|k \\ 0 & \text{otherwise} \end{cases}$

Q7 : The expected value of $\sum_{e: v \in e} x_e'$ is $\sum_{e: v \in e} k x_i / 2|V|k = \frac{k}{2|V|k} \sum_{e: v \in e} x_i \le \frac{k}{2|V|}$

By Markov's inequality, $\Pr[\sum_{e: v \in e} x_e' > k = 2|V| \cdot \mathrm{EXP}] = \frac{1}{2|V|}$.

Probability that each constraint is not satisfied is $\le \frac{1}{2|V|}$

Probability that 'some' constraints are not satisfied is $\le \sum_v \frac{1}{2|V|} = \frac{1}{2}$.

Q8 : Expected value of $\sum_e x_e'$ is $\sum_e k \cdot \frac{x_e}{2|V|k} = \frac{1}{2|V|} * \sum_e x_e$

$= \frac{1}{2|V|} \cdot OPT_{LP} \ge \frac{1}{2|V|} \cdot OPT$

Approximation ratio is $\frac{1}{2|V|}$.

Q9 : $S = \{1, 2\}$ $OPT = 2$

Q10 : Maximize $x_1 + x_2 + x_3 + x_4$

$x_1 + x_2 + x_3 \ge 1$

$x_1 + x_2 + x_4 \ge 1$

$x_1 + x_3 + x_4 \ge 1$

$x_2 + x_3 + x_4 \ge 1$

Q 11 : $x_1 = x_2 = x_3 = x_4 = 1/3$

Q 12 : 76

Q 13 : $\{1, 2\}$

Lecture Note 6

Q1 Input: $\mathcal{A}$

Output: $\mathcal{B}$

Constraint: 1) $\mathcal{C}$

2) $\mathcal{D} \geq 0.5$ OPT

Q2 Input: $\mathcal{A}$

Output: $\mathcal{B}$

Constraint: 1) $\mathcal{C}$

2) $'1/\mathcal{D} \leq 2 \cdot \mathrm{OPT}'$

Q3 return Problem 2 ($\mathcal{A}$);

We know that $\mathrm{OPT}' = \frac{1}{\mathrm{OPT}}$ as minimizing $\mathcal{D}$ is equivalent to maximizing $1/\mathcal{D}$.

Constraint 2) of Q1 $\quad \boxed{\mathcal{D} \geq \frac{1}{2}\mathrm{OPT}'} = \frac{1\ \cancel{\mathrm{OPT}}}{2\ \mathrm{OPT}}$

$\boxed{2\ \mathrm{OPT} \geq \frac{1}{\mathcal{D}}}$ constraint 2) of Q2

Hence, Problem in Q2 is equivalent to Q1.

Q4 If we have a 2-approximation algorithm for Problem 2, by Q3, we then also have a 0.5-approximation algorithm for 0.5-approximation algorithm for the problem in Q1. As we do not have the 0.5-approximation algorithm, we do not have the 2-approximation algorithm.

Q5.

OPT_{P1}	OPT_{P3}	Guarantee by 2-approximation algorithm	Guarantee for P_1
			≥ 0
0	1	≤ 2	≥ 0
0.1	0.9	≤ 1.8	≥ 0
0.2	0.8	≤ 1.6	≥ 0
0.3	0.7	≤ 1.4	≥ 0
0.4	0.6	≤ 1.2	≥ 0
0.5	0.5	≤ 1	≥ 0
0.6	0.4	≤ 0.8	≥ 0.2
0.7	0.3	≤ 0.6	≥ 0.4
0.8	0.2	≤ 0.4	≥ 0.6
0.9	0.1	≤ 0.2	≥ 0.8
1	0	$\leq 0.$	≥ 1

Q6 When $OPT_{P_3} = 1-\varepsilon$ for small ε, we know that $SOL_{P_3} \leq \alpha'(1-\varepsilon)$.

When ε is small enough, $\alpha'(1-\varepsilon) \geq 1$. We have $SOL_{P_3} \geq 1$.

We then have $SOL_{P_1} \leq 0 < \alpha \cdot OPT_{P_1} = \alpha \cdot \varepsilon$ for any $\alpha > 0$.

Q7 ~~d~~ $d(i,j)$: distant between house i, j for all $1 \leq i, j \leq n$.

Assumption $d(i,j) \leq d(i,j') + d(j',j)$

k : # ward offices.

$S_D \subseteq \{1, \dots, n\}$: houses with disability.

Q8 Have to set ward offices $S \subseteq \{1, \dots, n\}$

~~Q9~~ Q10
$$D_i = \begin{cases} \min_{j \in S} d(i,j) & \text{if } i \notin S_D \\ 2 \min_{j \in S} d(i,j) & \text{otherwise} \end{cases}$$

Minimize $\max_i D_i$

Q9 $|S| = k$.

Q11 1) $|S| = k$

2) $\max_i D_i \leq 1.999 \cdot OPT$

Q12 return 1.9999.ApproxKCenterwithDisabilities $(\rho, k, \emptyset)$;

Q13 If we have an 1.9999-approximation algorithm for k-center with disabilities, we will have an 1.9999-approximation algorithm for k-center problem.

As we do not have 1.9999-approximation algorithm for k-center, we do not have an 1.9999-approximation algorithm for the problem with disabilities.

Lecture Note 7

Q1 2

Q2 8 $5 \to 2 \to 1 \to 4 \to 7 \to 8 \to 9 \to 6 \to 3$

1 3 5 7

Q3 7

Q4 In any Hamiltonian path, there must be at least one nodes in ~~{~~ 2,4,6,8 which takes 7 steps to reach. If we know where the teasure is, we can reach all of the nodes in just one step.

Q5 ~~$5 \to 2 \to 5 \to 4 \to 7 \to 8 \to 9 \to 6 \to 3 \to 6 \to 5 \to 4 \to 1$~~

$5 \to 4 \to 7 \to 8 \to 9 \to 6 \to 5 \to 2 \to 5 \to 4 \to 1 \to 4 \to 5 \to 6 \to 3$

~~If~~ Also, in this map, it is not possible to visit 2,4,6,8 in less than 7 steps.

Q6 Receive yearly pension. $40 \times 8 = 320$ Million yens.

Q7 $80 - 32 = 48$ Million yens.

Q8

Q9 $\frac{48}{320}$

Q10 Receive lump sum since the beginning, $\frac{80}{320}$

Q11 We should check which of $\frac{X}{4Y}$ and $\frac{Y}{X}$ is the larger.

If $\frac{X}{4Y}$ is larger, we should receive lump sum at the beginning.

Otherwise, we should receive the yearly pension. Competive ratio is then $\max\left\{\frac{X}{4Y}, \frac{Y}{X}\right\}$.

Q12 We spend money in the ski rental problem, while we receive money in this problem.

Q13 lump sum amount X, yearly pension amount Y

prob. that we die at age i, p_i

Q14 Which year we will turn to receive yearly pension, j

Q15 ~~$\forall$~~ $60 \le j \le 99$

Q16 Maximize $\sum_{i} p_i R_i$ when $R_i = \begin{cases} iY & \text{if } i<j \\ 80 - iY & \text{otherwise} \end{cases}$

Q17 We can try all possible j. The algorithm will complete in polynomial time.

Q18 We should receive yearly pension. Expected income is 160 Million yens.

Q19 We should use the second strategy if we have the probability distribution. Otherwise, we use the first strategy.

Lecture Notes

Q1 s_1 is always the largest among $\{s_1\}$ $\rightarrow$ chosen with prob. $1/n$

With random order, s_1 is the largest with prob. $1/n$. $\rightarrow$ we correctly choose with prob. $1/n^2$

Q2 We consider s_2 with prob. $1 - \underbrace{1/n}_{\text{prob that we choose } s_1}$.

s_2 is the largest among $\{s_1, s_2\}$ with prob. $1/2$ $\rightarrow$ we then choose it with prob. $\frac{2}{n}$.

$\therefore$ We choose s_2 with prob. $\frac{n-1}{n} \cdot 1/2 \cdot \frac{2}{n} = \frac{n-1}{n^2}$.

Prob. that s_2 is the largest when $s_2 > s_1$ is $\frac{2}{n}$.

Hence, we correctly choose s_2 with prob. $\frac{2}{n} \cdot \frac{n-1}{n^2}$

Q3 We consider s_3 with prob. $1 - 1/n - \frac{n-1}{n^2} = \frac{(n-1)^2}{n^3}$.

s_3 is the largest among $\{s_1, s_2, s_3\}$ with prob. $1/3$ $\rightarrow$ we then choose it with prob. $3/n$.

$\therefore$ We choose s_3 with prob. $\frac{(n-1)^2}{n^3} \cdot 1/3 \cdot 3/n = \frac{(n-1)^2}{n^3}$.

Prob. that s_3 is the largest when it is largest among $\{s_1, s_2, s_3\}$ is $3/n$:

Hence, we correctly choose s_3 with prob. $3/n \cdot \frac{(n-1)^2}{n^3}$

Q4 $\sum_{i=1}^{n} \frac{(n-1)^{i-1}}{n^i} \cdot \frac{i}{n}$

Q5 $= \frac{1}{n(n-1)} \sum_{i=1}^{n} \frac{(n-1)^i}{n^i} \cdot i$

$= \frac{1}{n\cancel{(n-1)}} \cdot \cancel{(n-1)}\, n^{1-n} [n^n - 2(n-1)^n] = n^{-n}[n^n - 2(n-1)^n] = 1 - 2\left(\frac{n-1}{n}\right)^n$.

Q6 Success prob. $= 1 - \frac{2}{e}$ $\rightarrow$ less than the success probability of the strategy in the class which is $1/e$

Q7 0.3668

Q8
100 – 0.3668
1000 – 0.3611
10000 – 0.3694

Q9 It is always around 0.36, but there is no sightable trend.